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# IR-Improved DGLAP Theory<sup>†</sup>

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## Abstract

We show that it is possible to improve the infrared aspects of the standard treatment of the DGLAP evolution theory to take into account a large class of higher order corrections that significantly improve the precision of the theory for any given level of fixed-order calculation of its respective kernels. We illustrate the size of the effects we resum using the moments of the parton distributions.

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In the preparation of the physics for the precision QCD×EW(electroweak) [1,2] LHC physics studies, all aspects of the calculation of the cross sections and distributions for the would-be physical observables must be re-examined if precision tags such as that envisioned for the luminosity theoretical precision are to be realized, i.e., 1% cross section predictions for single heavy gauge boson production in 14 TeV pp collisions when that heavy gauge boson decays into a light lepton pair. The QCD DGLAP [3] evolution of the structure functions from the typical reference scale of data input,  $\mu_0 \sim 1 - 2 GeV$ , to the respective hard scale is one step that warrants further study, as it is well-known to many. Many authors [4-7] have provided excellent realizations of this evolution in the recent literature. Here, we will re-examine the infrared aspects of the basic DGLAP theory itself to try to improve the treatment to a level consistent with the new era of precision QCD×EW physics needed for the LHC physics objectives.

Specifically, the motivation for the improvement which we develop can be seen already in the basic results in Refs. [3] for the kernels that determine the evolution of the structure functions by the attendant DGLAP evolution of the corresponding parton densities by the standard methodology. Consider the evolution of the non-singlet(NS) parton density function  $q^{NS}(x)$ , where  $x$  can be identified as Bjorken's variable as usual. The basic starting point of our analysis is the infrared divergence in the kernel that determines this evolution:

$$\frac{dq^{NS}(x, t)}{dt} = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} q^{NS}(y, t) P_{qq}(x/y) \quad (1)$$

where the well-known result for the kernel  $P_{qq}(z)$  is, for  $z < 1$ ,

$$P_{qq}(z) = C_F \frac{1+z^2}{1-z} \quad (2)$$

when we set  $t = \ln \mu^2 / \mu_0^2$  for some reference scale  $\mu_0$  with which we study evolution to the scale of interest  $\mu$ .<sup>1</sup> Here,  $C_F = (N_c^2 - 1)/(2N_c)$  is the quark color representation's quadratic Casimir invariant where  $N_c$  is the number of colors and so that it is just 3. This kernel has an unintegrable IR singularity at  $z = 1$ , which is the point of zero energy gluon emission and this is as it should be. The standard treatment of this very physical effect is to regularize it by the replacement

$$\frac{1}{(1-z)} \rightarrow \frac{1}{(1-z)_+} \quad (3)$$

with the distribution  $\frac{1}{(1-z)_+}$  defined so that for any suitable test function  $f(z)$  we have

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z) - f(1)}{(1-z)}. \quad (4)$$

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<sup>1</sup>We will generally follow Ref. [8] and set  $\mu_0 = \Lambda_{QCD}$  without loss of content since  $dt = dt'$  when  $t = \ln \mu^2 / \Lambda_{QCD}^2$ ,  $t' = \ln \mu^2 / \mu_0^2$  for fixed values of  $\Lambda_{QCD}, \mu_0$ .

A possible representation of  $1/(1-z)_+$  is seen to be

$$\frac{1}{(1-z)_+} = \frac{1}{(1-z)}\theta(1-\epsilon-z) + \ln \epsilon \delta(1-z) \quad (5)$$

with the understanding that  $\epsilon \downarrow 0$ . We use the notation  $\theta(x)$  for the step function from 0 for  $x < 0$  to 1 for  $x \geq 0$  and  $\delta(x)$  is Dirac's delta function. The final result for  $P_{qq}(z)$  is then obtained by imposing the physical requirement [3] that

$$\int_0^1 dz P_{qq}(z) = 0, \quad (6)$$

which is satisfied by adding the effects of virtual corrections at  $z = 1$  so that finally

$$P_{qq}(z) = C_F \left( \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right). \quad (7)$$

The smooth behavior in the original real emission result from the Feynman rules, with a divergent  $1/(1-z)$  behavior as  $z \rightarrow 1$ , has been replaced with a mathematical artifact: the regime  $1-\epsilon < z < 1$  now has no probability at all and at  $z = 1$  we have a large negative integrable contribution so that we end-up finally with a finite (zero) value for the total integral of  $P_{qq}(z)$ . This mathematical artifact is what we wish to improve here; for, in the precision studies of Z physics [9–11] at LEP1, it has been found that such mathematical artifacts can indeed impair the precision tag which one can achieve with a given fixed order of perturbation theory. An analogous case is now well-known in the theory of QCD higher order corrections, where the FNAL data on  $p_T$  spectra clearly show the need for improvement of fixed-order results by resumming large logs associated with soft gluons [12, 13]. For reference, note that at the LHC, 2 TeV partons are realistic so that  $z \cong 0.001$  means  $\sim 2-3$  GeV soft gluons, which are clearly above the LHC detector thresholds, in complete analogy with the situation at LEP where  $z \cong 0.001$  meant  $\sim 100$  MeV photons which were also above the LEP detector thresholds – just as resummation was necessary to describe this view of the LEP data, so too we may argue it will be necessary to describe the LHC data on the corresponding view. And, more importantly, why should we have to set  $P_{qq}(z)$  to 0 for  $1-\epsilon < z < 1$  when it actually has its largest values in this very regime?

By mathematical artifact we do not mean that there is an error in the computations that lead to it; indeed, it is well-known that this  $+$ -function behavior is exactly what one gets at  $\mathcal{O}(\alpha_s)$  for the bremsstrahlung process. The artifact is that the behavior of the differential spectrum of the process for  $z \rightarrow 1$  in  $\mathcal{O}(\alpha_s)$  is unintegrable and has to be cut-off and thus this spectrum is only poorly represented by the  $\mathcal{O}(\alpha_s)$  calculation; for, the resummation of the large soft higher order effects as we present below changes the  $z \rightarrow 1$  behavior non-trivially, as from our resummation we will find that the  $\frac{1}{1-z}$ -behavior is modified to  $(1-z)^{\gamma-1}$ ,  $\gamma > 0$ . This is a testable effect, as we have seen in its QED analogs in Z physics at LEP1 [9–11]: if the experimentalist measures the cross section

for bremsstrahlung for gluons(photons) down to energy fraction  $\epsilon_0$ ,  $\epsilon_0 > 0$ , in our new resummed theory presented below, the result will approach a finite value from below as  $-\epsilon_0^\gamma$  whereas the  $\mathcal{O}(\alpha_s)$  +-function prediction would increase without limit as  $-\ln \epsilon_0$ . The exponentiated result has been verified by the data at LEP1.

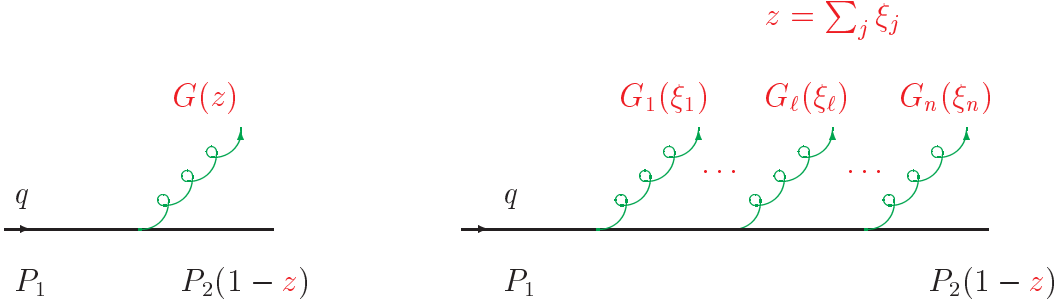
The important point is that the traditional resummations in N-moment space for the DGLAP kernels address only the short-distance contributions to their higher order corrections. The deep question we deal with in this paper concerns, then, how much of the complete soft limit of the DGLAP kernels is contained in the anomalous dimensions of the leading twist operators in Wilson's expansion, an expansion which resides on the very tip of the light-cone? Are all of the effects of the very soft gluon emission, involving, as they most certainly do, arbitrarily long wavelength quanta, representable by the physics at the tip of the light-cone? The Heisenberg uncertainty principle surely tells us that answer can not be affirmative. In this paper, we calculate these long-wavelength gluon effects on the DGLAP kernels that are not included (see the discussion below) in the standard treatment of Wilson's expansion. We therefore do not contradict the results of the large N-moment space resummations such as that presented in Ref. [14] nor do we contradict the renormalon chain-type resummation as done in Ref. [15].

We employ the exact re-arrangement of the Feynman series for QCD as it has been shown in Ref. [16, 17]. For completeness, as this QCD exponentiation theory is not generally familiar, we reproduce its essential aspects in our Appendix. The idea is to sum up the leading IR terms in the corrections to  $P_{qq}$  with the goal that they will render integrable the IR singularity that we have in its lowest order form. This will remove the need for mathematical artifacts and exhibit more accurately the true predictions of the full QCD theory in terms of fully physical results.

We apply the QCD exponentiation master formula in eq.(80) in our Appendix (see also Ref. [16]), following the analogous discussion then for QED in Refs. [10, 11], to the gluon emission transition that corresponds to  $P_{qq}(z)$ , i.e., to the squared amplitude for  $q \rightarrow q(z) + G(1-z)$  so that in the Appendix one replaces everywhere the squared amplitudes for the  $\bar{Q}'Q \rightarrow \bar{Q}'''Q''$  processes with those for the former one plus its  $nG$  analoga with the attendant changes in the phase space and kinematics dictated by the standard methods; this implies that in eq.(53) of the first paper in Ref. [3] we have from eq.(80) the replacement ( see Fig. )

$$\begin{aligned}
P_{BA} &= P_{BA}^0 \equiv \frac{1}{2}z(1-z) \overline{\sum_{spins}} \frac{|V_{A \rightarrow B+C}|^2}{p_\perp^2} \\
&\Rightarrow \\
P_{BA} &= \frac{1}{2}z(1-z) \overline{\sum_{spins}} \frac{|V_{A \rightarrow B+C}|^2}{p_\perp^2} z^{\gamma_q} F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q}
\end{aligned} \tag{8}$$

where  $A = q$ ,  $B = G$ ,  $C = q$  and  $V_{A \rightarrow B+C}$  is the lowest order amplitude for  $q \rightarrow$



(a) (b)  
Figure 1: In (a), we show the usual process  $q \rightarrow q(1-z) + G(z)$ ; in (b), we show its multiple gluon improvement  $q \rightarrow q(1-z) + G_1(\xi_1) + \cdots + G_n(\xi_n)$ ,  $z = \sum_j \xi_j$ .

$G(z) + q(1-z)$ , so that we get the un-normalized exponentiated result

$$P_{qq}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \frac{1+z^2}{1-z} (1-z)^{\gamma_q} \quad (9)$$

where [10, 11, 16, 17]

$$\gamma_q = C_F \frac{\alpha_s}{\pi} t = \frac{4C_F}{\beta_0} \quad (10)$$

$$\delta_q = \frac{\gamma_q}{2} + \frac{\alpha_s C_F}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right) \quad (11)$$

and

$$F_{YFS}(\gamma_q) = \frac{e^{-C_E \gamma_q}}{\Gamma(1 + \gamma_q)}. \quad (12)$$

Here,

$$\beta_0 = 11 - \frac{2}{3}n_f$$

, where  $n_f$  is the number of active quark flavors,

$$C_E = .5772 \dots$$

is Euler's constant and  $\Gamma(w)$  is Euler's gamma function. The function  $F_{YFS}(z)$  was already introduced by Yennie, Frautschi and Suura [18] in their analysis of the IR behavior of QED.

We see immediately that the exponentiation has removed the unintegrable IR divergence at  $z = 1$ . For reference, we note that we have in (9) resummed the terms<sup>2</sup>  $\mathcal{O}(\ln^k(1 - z)t^\ell\alpha_s^n)$ ,  $n \geq \ell \geq k$ , which originate in the IR regime and which exponentiate. The important point is that we have not dropped outright the terms that do not exponentiate but have organized them into the residuals  $\tilde{\beta}_m$  in the analog of eq.(80). The application of eq.(80) to obtain eq.(9) proceeds as follows. First, the exponent in the exponential factor in front of the expression on the RHS of eq.(80) is readily seen to be from eq.(77), using the well-known results for the respective real and virtual infrared functions from Refs. [16, 17],

$$\begin{aligned} SUM_{IR}(QCD) &= 2\alpha_s ReB_{QCD} + 2\alpha_s \tilde{B}_{QCD}(K_{max}) \\ &= \frac{1}{2} \left( 2C_F \frac{\alpha_s}{\pi} t \ln \frac{K_{max}}{E} + C_F \frac{\alpha_s}{2\pi} t + \frac{\alpha_s C_F}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right) \right) \end{aligned} \quad (13)$$

where on the RHS of the last result we have already applied the DGLAP synthesization procedure in the third paper in Ref. [16] to remove the collinear singularities,  $\ln \Lambda_{QCD}^2/m_q^2 - 1$ , in accordance with the standard QCD factorization theorems [19]. This means that, identifying the LHS of eq.(80) as the sum over final states and average over initial states of the respective process divided by the incident flux and replacing that incident flux by the respective initial state density according to the standard methods for the process  $q \rightarrow q(1 - z) + G(z)$ , occurring in the context of a hard scattering at scale  $Q$  as it is for eq.(53) in the first paper in Ref. [3], the soft gluon effects for energy fraction  $< z \equiv K_{max}/E$  give the result, from eq.(80), that, working through to the  $\tilde{\beta}_1$ -level and using  $q_2$  to represent the momentum conservation via the other degrees of freedom for the attendant hard process,

$$\begin{aligned} \int \frac{\alpha_s(t)}{2\pi} P_{BA} dt dz &= e^{SUM_{IR}(QCD)(z)} \int \{ \tilde{\beta}_0 \int \frac{d^4 y}{(2\pi)^4} e^{\{iy \cdot (p_1 - p_2) + \int^{k < K_{max}} \frac{d^3 k}{k} \tilde{S}_{QCD}(k) [e^{-iy \cdot k} - 1]\}} \\ &+ \int \frac{d^3 k_1}{k_1} \tilde{\beta}_1(k_1) \int \frac{d^4 y}{(2\pi)^4} e^{\{iy \cdot (p_1 - p_2 - k_1) + \int^{k < K_{max}} \frac{d^3 k}{k} \tilde{S}_{QCD}(k) [e^{-iy \cdot k} - 1]\}} \\ &+ \dots \} \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0} \\ &= e^{SUM_{IR}(QCD)(z)} \int \{ \tilde{\beta}_0 \int_{-\infty}^{\infty} \frac{dy}{(2\pi)} e^{\{iy \cdot (E_1 - E_2) + \int^{k < K_{max}} \frac{d^3 k}{k} \tilde{S}_{QCD}(k) [e^{-iy \cdot k} - 1]\}} \\ &+ \int \frac{d^3 k_1}{k_1} \tilde{\beta}_1(k_1) \int_{-\infty}^{\infty} \frac{dy}{(2\pi)} e^{\{iy \cdot (E_1 - E_2 - k_1^0) + \int^{k < K_{max}} \frac{d^3 k}{k} \tilde{S}_{QCD}(k) [e^{-iy \cdot k} - 1]\}} \\ &+ \dots \} \frac{d^3 p_2}{p_2^0 q_2^0} \end{aligned} \quad (14)$$

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<sup>2</sup>Following the standard LEP Yellow Book [9] convention, we do not include the order of the first nonzero term in counting the order of its higher order corrections.

where we set  $E_i = p_i^0$ ,  $i = 1, 2$  and the real infrared function  $\tilde{S}_{\text{QCD}}(k)$  is well-known as well:

$$\tilde{S}_{\text{QCD}}(k) = -\frac{\alpha_s C_F}{8\pi^2} \left( \frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2 \Big|_{\text{DGLAP synthesized}} \quad (15)$$

and we indicate as above that the DGLAP synthesization procedure in Refs. [16] is to be applied to its evaluation to remove its collinear singularities; we are using the kinematics of the first paper in Ref. [3] in their computation of  $P_{BA}(z)$  in their eq.(53), so that the relevant value of  $k_\perp^2$  is indeed  $Q^2$ . It means that the computation can also be seen to correspond to computing the IR function for the standard t-channel kinematics and taking  $\frac{1}{2}$  of the result to match the single line emission in  $P_{Gq}$ . The two important integrals needed in (14) were already studied in Ref. [18]:

$$\begin{aligned} I_{YFS}(zE, 0) &= \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{[iy(zE) + \int^{k < zE} \frac{d^3k}{k} \tilde{S}_{QCD}(k)(e^{-iyk} - 1)]} \\ &= F_{YFS}(\gamma_q) \frac{\gamma_q}{zE} \\ I_{YFS}(zE, k_1) &= \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{[iy(zE - k_1) + \int^{k < zE} \frac{d^3k}{k} \tilde{S}_{QCD}(k)(e^{-iyk} - 1)]} \\ &= \left( \frac{zE}{zE - k_1} \right)^{1-\gamma_q} I_{YFS}(zE, 0) \end{aligned} \quad (16)$$

When we introduce the results in (16) into (14) we can identify the factor

$$\int \left( \tilde{\beta}_0 \frac{\gamma_q}{zE} + \int dk_1 k_1 d\Omega_1 \tilde{\beta}_1(k_1) \left( \frac{zE}{zE - k_1} \right)^{1-\gamma_q} \frac{\gamma_q}{zE} \right) \frac{d^3p_2}{E_2 q_2^0} = \int dt \frac{\alpha_s(t)}{2\pi} P_{BA}^0 dz + \mathcal{O}(\alpha_s^2). \quad (17)$$

where  $P_{BA}^0$  is the unexponentiated result in the first line of (8). This leads us finally to the exponentiated result in the second line of (8) by elementary differentiation:

$$P_{BA} = P_{BA}^0 z^{\gamma_q} F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \quad (18)$$

Here, we also may note how one can see that the terms we exponentiate are not included in the standard treatment of Wilson's expansion: From the standard method [20], the N-th moment of the invariants  $T_{i,\ell}$ ,  $i = L, 2, 3$ ,  $\ell = q, G$ , of the forward Compton amplitude in DIS is projected by

$$\mathcal{P}_N \equiv \left[ \frac{q^{\{\mu_1 \dots \mu_N\}}}{N!} \frac{\partial^N}{\partial p^{\mu_1} \dots \partial p^{\mu_N}} \right] \Big|_{p=0} \quad (19)$$

where  $x_{Bj} = Q^2/(2qp)$  in the standard DIS notation; this projects the coefficient of  $1/(2x_{Bj})^N$ . For the dominant terms which we resum here, the characteristic behavior would correspond formally to  $\gamma_q$ -dependent anomalous dimensions associated with the respective coefficient whereas by definition Wilson's expansion does not contain such. In

more phenomenologically familiar language, it is well-known that the parton model used in this paper to calculate the large distance effects that improve the kernels contains such effects whereas Wilson's expansion does not: for example, the parton model can be used for Drell-Yan processes, Wilson's expansion can not. Similarly, any Wilson-expansion guided procedure used to infer the kernels via inverse Mellin transformation, by calculating the coefficient of  $(1/z)^n$  in Wilson's expansion, will necessarily omit the dominant IR terms which we resum. Here, we stress that we refer to the properties of the expansion of the invariant functions  $T_i$ , *not to the expansion of the kernels themselves*, as the latter are related to the respective anomalous dimension matrix elements by inverse Mellin transformations.

The normalization condition in eq.(6) then gives us the final expression

$$P_{qq}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \left[ \frac{1+z^2}{1-z} (1-z)^{\gamma_q} - f_q(\gamma_q) \delta(1-z) \right] \quad (20)$$

where

$$f_q(\gamma_q) = \frac{2}{\gamma_q} - \frac{2}{\gamma_q + 1} + \frac{1}{\gamma_q + 2}. \quad (21)$$

The latter result is then our IR-improved kernel for NS DGLAP evolution in QCD. We note that the appearance of the integrable function  $(1-z)^{-1+\gamma_q}$  in the place of  $\frac{1}{(1-z)_+}$  was already anticipated by Gribov and Lipatov in Refs. [3]. Here, we have calculated the value of  $\gamma_q$  in a systematic rearrangement of the QCD perturbation theory that allows one to work to any exact order in the theory without dropping any part of the theory's perturbation series.

The standard DGLAP theory tells us that the kernel  $P_{Gq}(z)$  is related to  $P_{qq}(1-z)$  directly: for  $z < 1$ , we have

$$P_{Gq}(z) = P_{qq}(1-z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \frac{1+(1-z)^2}{z} z^{\gamma_q}. \quad (22)$$

This then brings us to our first non-trivial check of the new IR-improved theory; for, the conservation of momentum tells us that

$$\int_0^1 dz z (P_{Gq}(z) + P_{qq}(z)) = 0. \quad (23)$$

Using the new results in eqs.(20,22), we have to check that the following integral vanishes:

$$I = \int_0^1 dz z \left( \frac{1+(1-z)^2}{z} z^{\gamma_q} + \frac{1+z^2}{1-z} (1-z)^{\gamma_q} - f_q(\gamma_q) \delta(1-z) \right). \quad (24)$$

To see that it does, note that

$$\frac{z}{1-z} = \frac{z-1+1}{1-z} = -1 + \frac{1}{1-z}. \quad (25)$$



Introducing this result into eq.(24) we get

$$I = \int_0^1 dz \{ (1 + (1-z)^2)z^{\gamma_q} - (1+z^2)(1-z)^{\gamma_q} + \frac{1+z^2}{1-z}(1-z)^{\gamma_q} - f_q(\gamma_q)\delta(1-z) \}. \quad (26)$$

The integrals over the first two terms on the right-hand side (RHS) of (26) exactly cancel as one sees by using the change of variable  $z \rightarrow 1-z$  in one of them and the integral over the last two terms on the RHS of (26) vanishes from the normalization in eq.(6). Thus we conclude that

$$I = 0. \quad (27)$$

The quark momentum sum rule is indeed satisfied.

Having improved the IR divergence properties of  $P_{qq}(z)$  and  $P_{Gq}(z)$ , we now turn to  $P_{GG}(z)$  and  $P_{qG}(z)$ . We first note that the standard formula for  $P_{qG}(z)$ ,

$$P_{qG}(z) = \frac{1}{2}(z^2 + (1-z)^2), \quad (28)$$

is already well-behaved (integrable) in the IR regime. Thus, we do not need to improve it here to make it integrable and we note that the singular contributions in the other kernels are expected to dominate the evolution effects in any case. We do not exclude improving it for the best precision [21] and we return to this point presently.

This brings us then to  $P_{GG}(z)$ . Its lowest order form is

$$P_{GG}(z) = 2C_G \left( \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right) \quad (29)$$

which again exhibits unintegrable IR singularities at both  $z = 1$  and  $z = 0$ . (Here,  $C_G$  is the gluon quadratic Casimir invariant, so that it is just  $N_c = 3$ .) If we repeat the QCD exponentiation calculation carried-out above by using the color representation for the gluon rather than that for the quarks, i.e., if we apply the exponentiation analysis in the Appendix to the squared amplitude for the process  $G \rightarrow G(z) + G(1-z)$ , we get the exponentiated un-normalized result

$$P_{GG}(z) = 2C_G F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} \left( \frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z} (1-z)^{\gamma_G} + \frac{1}{2} (z^{1+\gamma_G} (1-z) + z(1-z)^{1+\gamma_G}) \right) \quad (30)$$

wherein we obtain the  $\gamma_G$  and  $\delta_G$  from the expressions for  $\gamma_q$  and  $\delta_q$  by the substitution  $C_F \rightarrow C_G$ :

$$\gamma_G = C_G \frac{\alpha_s}{\pi} t = \frac{4C_G}{\beta_0} \quad (31)$$

$$\delta_G = \frac{\gamma_G}{2} + \frac{\alpha_s C_G}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right). \quad (32)$$

We see again that exponentiation has again made the singularities at  $z = 1$  and  $z = 0$  integrable.

To normalize  $P_{GG}$ , we take into account the virtual corrections such that the gluon momentum sum rule

$$\int_0^1 dz z (2n_f P_{qG}(z) + P_{GG}(z)) = 0 \quad (33)$$

is satisfied. This gives us finally the IR-improved result

$$\begin{aligned} P_{GG}(z) = 2C_G F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} & \left\{ \frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z} (1-z)^{\gamma_G} \right. \\ & \left. + \frac{1}{2} (z^{1+\gamma_G} (1-z) + z(1-z)^{1+\gamma_G}) - f_G(\gamma_G) \delta(1-z) \right\} \end{aligned} \quad (34)$$

where for  $f_G(\gamma_G)$  we get

$$\begin{aligned} f_G(\gamma_G) = \frac{n_f}{6C_G F_{YFS}(\gamma_G)} e^{-\frac{1}{2}\delta_G} & + \frac{2}{\gamma_G(1+\gamma_G)(2+\gamma_G)} + \frac{1}{(1+\gamma_G)(2+\gamma_G)} \\ & + \frac{1}{2(3+\gamma_G)(4+\gamma_G)} + \frac{1}{(2+\gamma_G)(3+\gamma_G)(4+\gamma_G)}. \end{aligned} \quad (35)$$

It is these improved results in eqs.(20,22,34) for  $P_{qq}(z)$ ,  $P_{qG}(z)$  and  $P_{GG}(z)$  that we use together with the standard result in (29) for  $P_{qG}(z)$  as the IR-improved DGLAP theory.

For clarity we summarize at this point the new IR-improved kernel set as follows:

$$P_{qq}^{exp}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \left[ \frac{1+z^2}{1-z} (1-z)^{\gamma_q} - f_q(\gamma_q) \delta(1-z) \right], \quad (36)$$

$$P_{Gq}^{exp}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \frac{1+(1-z)^2}{z} z^{\gamma_q}, \quad (37)$$

$$\begin{aligned} P_{GG}^{exp}(z) = 2C_G F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} & \left\{ \frac{1-z}{z} z^{\gamma_G} + \frac{z}{1-z} (1-z)^{\gamma_G} \right. \\ & \left. + \frac{1}{2} (z^{1+\gamma_G} (1-z) + z(1-z)^{1+\gamma_G}) - f_G(\gamma_G) \delta(1-z) \right\}, \end{aligned} \quad (38)$$

$$P_{qG}(z) = \frac{1}{2} (z^2 + (1-z)^2), \quad (39)$$

where we have introduced the superscript *exp* to denote the exponentiated results henceforth.

Returning now to the improvement of  $P_{qG}(z)$ , let us record it as well for the sake of completeness and of providing better precision. Applying eq.(80) to the process  $G \rightarrow q + \bar{q}$ , we get the exponentiated result

$$P_{qG}^{exp}(z) = F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} \frac{1}{2} \{ z^2 (1-z)^{\gamma_G} + (1-z)^2 z^{\gamma_G} \}. \quad (40)$$

The gluon momentum sum rule then gives the new normalization constant for the  $P_{GG}^{exp}$  via the result

$$\begin{aligned} \bar{f}_G(\gamma_G) = \frac{n_f}{C_G} & \frac{1}{(1+\gamma_G)(2+\gamma_G)(3+\gamma_G)} + \frac{2}{\gamma_G(1+\gamma_G)(2+\gamma_G)} + \frac{1}{(1+\gamma_G)(2+\gamma_G)} \\ & + \frac{1}{2(3+\gamma_G)(4+\gamma_G)} + \frac{1}{(2+\gamma_G)(3+\gamma_G)(4+\gamma_G)}. \end{aligned} \quad (41)$$

The constant  $\bar{f}_G$  should be substituted for  $f_G$  in  $P_{GG}^{exp}$  whenever the exponentiated result in (40) is used. These results (39), (40), and (41) are our new improved DGLAP kernel set, with the option exponentiating  $P_{qG}$  as well. Let us now look into their effects on the moments of the structure functions by discussing the corresponding effects on the moments of the parton distributions.

We know that moments of the kernels determine the exponents in the logarithmic variation [3, 22] of the moments of the quark distributions and, thereby, of the moments of the structure functions themselves. To wit, in the non-singlet case, we have

$$\frac{dM_n^{NS}(t)}{dt} = \frac{\alpha_s(t)}{2\pi} A_n^{NS} M_n^{NS}(t) \quad (42)$$

where

$$M_n^{NS}(t) = \int_0^1 dz z^{n-1} q^{NS}(z, t) \quad (43)$$

and the quantity  $A_n^{NS}$  is given by

$$\begin{aligned} A_n^{NS} &= \int_0^1 dz z^{n-1} P_{qq}^{exp}(z), \\ &= C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} [B(n, \gamma_q) + B(n+2, \gamma_q) - f_q(\gamma_q)] \end{aligned} \quad (44)$$

where  $B(x, y)$  is the beta function given by

$$B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

. This should be compared to the un-IR-improved result [3, 22]:

$$A_n^{NSo} \equiv C_F \left[ -\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=2}^n \frac{1}{j} \right]. \quad (45)$$

The asymptotic behavior for large  $n$  is now very different, as the IR-improved exponent approaches a constant, a multiple of  $-f_q$ , as we would expect as  $n \rightarrow \infty$  because  $\lim_{n \rightarrow \infty} z^{n-1} = 0$  for  $0 \leq z < 1$  whereas, as it is well-known, the un-IR-improved result in (45) diverges as  $-2C_F \ln n$  as  $n \rightarrow \infty$ . The two results are also different at finite  $n$ : for  $n = 2$  we get, for example, for  $\alpha_s \cong .118$  [23],

$$A_2^{NS} = \begin{cases} C_F(-1.33) & , \quad \text{un-IR-improved} \\ C_F(-0.966) & , \quad \text{IR-improved} \end{cases} \quad (46)$$

so that the effects we have calculated are important for all  $n$  in general. For completeness, we note that the solution to (42) is given by the standard methods as

$$\begin{aligned} M_n^{NS}(t) &= M_n^{NS}(t_0) e^{\int_{t_0}^t dt' \frac{\alpha_s(t')}{2\pi} A_n^{NS}(t')} \\ &= M_n^{NS}(t_0) e^{\bar{a}_n [Ei(\frac{1}{2}\delta_1 \alpha_s(t_0)) - Ei(\frac{1}{2}\delta_1 \alpha_s(t))]} \\ &\xRightarrow[t, t_0 \text{ large with } t \gg t_0]{} M_n^{NS}(t_0) \left( \frac{\alpha_s(t_0)}{\alpha_s(t)} \right)^{\bar{a}'_n} \end{aligned} \quad (47)$$

where  $Ei(x) = \int_{-\infty}^x dr e^r/r$  is the exponential integral function,

$$\begin{aligned}\bar{a}_n &= \frac{2C_F}{\beta_0} F_{YFS}(\gamma_q) e^{\frac{\gamma_q}{4}} [B(n, \gamma_q) + B(n+2, \gamma_q) - f_q(\gamma_q)] \\ \bar{a}'_n &= \bar{a}_n \left( 1 + \frac{\delta_1}{2} \frac{(\alpha_s(t_0) - \alpha_s(t))}{\ln(\alpha_s(t_0)/\alpha_s(t))} \right)\end{aligned}\tag{48}$$

with

$$\delta_1 = \frac{C_F}{\pi} \left( \frac{\pi^2}{3} - \frac{1}{2} \right)$$

. We can compare with the un-IR-improved result in which the last line in eq.(47) holds exactly with  $\bar{a}'_n = 2A_n^{NS^o}/\beta_0$ . Phenomenologically, for  $n = 2$ , taking  $Q_0 = 2\text{GeV}$  and evolving to  $Q = 100\text{GeV}$ , if we set  $\Lambda_{QCD} \cong .2\text{GeV}$  and use  $n_f = 5$  for definiteness of illustration, we see from eqs. (47,48) that we get a shift of the respective evolved NS moment by  $\sim 5\%$ , which is of some interest in view of the expected HERA precision [24].

We give now the remaining elements of the anomalous dimension matrix in its 'best' IR-improved form for completeness:

$$A_n^{Gq} = \int_0^1 dz z^{n-1} P_{Gq}^{exp}(z) = C_F F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \left[ \frac{1}{n + \gamma_q - 1} + B(3, n + \gamma_q - 1) \right], \tag{49}$$

$$\begin{aligned}A_n^{GG} &= \int_0^1 dz z^{n-1} P_{GG}^{exp}(z) = 2C_G F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} \{ B(n+1, \gamma_G) + B(n + \gamma_G - 1, 2) \\ &\quad + \frac{1}{2}(B(n+1, \gamma_G + 2) + B(n + \gamma_G + 1, 2)) - \bar{f}(\gamma_G) \},\end{aligned}\tag{50}$$

$$2n_f A_n^{qG} = 2n_f \int_0^1 dz z^{n-1} P_{qG}^{exp}(z) = 2T(F) F_{YFS}(\gamma_G) e^{\frac{1}{2}\delta_G} (B(n+2, 1 + \gamma_G) + B(n + \gamma_G, 3)), \tag{51}$$

where  $T(F) = \frac{1}{2}n_f$ . We note that the un-exponentiated value of the last result in eq.(51) is a well-known one [3, 22],  $2T(F) \frac{2+n+n^2}{n(n+1)(n+2)}$ , and it would be used whenever we do not choose to exponentiate  $P_{qG}$ . We will investigate the further implications of these IR-improved results for LHC physics elsewhere [21].

In the discussion so far, we have used the lowest order DGLAP kernel set to illustrate how important the resummation which we present here can be. In the literature [25, 26], there are now exact results up to  $\mathcal{O}(\alpha_s^3)$  for the DGLAP kernels. The question naturally arises as to the relationship of our work to these fixed-order exact results. We stress first that we are presenting an improvement of the fixed-order results such that the singular pieces of the any exact fixed-order result, i.e., the  $\frac{1}{(1-z)_+}$  parts, are exponentiated so that they are replaced with integrable functions proportional to  $(1-z)^{\gamma-1}$  with  $\gamma$  positive as we have illustrated above. Since the series of logs which we resum to accomplish this has the structure  $\alpha_s^\ell t^\ell \ln^n(1-z)$ ,  $\ell \geq n$  these terms are not already present in the results in Refs. [25, 26]. As we use the formula in eq.(80), there will be no double counting if

we implement our IR-improvement of the exact fixed-order results in Refs. [25, 26]. The detailed discussion of the application of our theory to the results in Refs. [25, 26] will appear elsewhere [21]. For reference, we note that the higher order kernel corrections in Refs. [25, 26] are perturbatively related to the leading order kernels, so one can expect that the size of the exponentiation effects illustrated above will only be perturbatively modified by the higher order kernel corrections, leaving the same qualitative behavior in general.

In the interest of specifichness, let us illustrate the IR-improvement of  $P_{qq}$  when calculated to three loops using the results in Refs. [25, 26]. Considering the non-singlet case for definiteness (a similar analysis holds for the singlet case) we write in the notation of the latter references

$$P_{ns}^+ = P_{qq}^v + P_{q\bar{q}}^v \equiv \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} P_{ns}^{(n)+} \quad (52)$$

where at order  $\mathcal{O}(\alpha_s)$  we have

$$P_{ns}^{(0)+}(z) = 2C_F \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right\} \quad (53)$$

which shows that  $P_{ns}^{(0)+}(z)$  agrees with the unexponentiated result in (7) for  $P_{qq}$  except for an overall factor of 2. We use this latter identification to connect our work with that in Refs. [25, 26] in the standard methodology. In Refs. [25, 26], exact results are given for  $P_{ns}^{(1)+}(z)$ , and in Refs. [26] exact results are given for  $P_{ns}^{(2)+}(z)$ . When we apply the result in (80) to the squared amplitudes for the processes  $q \rightarrow q + X$ ,  $\bar{q} \rightarrow q + X'$ , we get the exponentiated result

$$\begin{aligned} P_{ns}^{+,exp}(z) = & \left(\frac{\alpha_s}{4\pi}\right) 2P_{qq}^{exp}(z) + F_{YFS}(\gamma_q) e^{\frac{1}{2}\delta_q} \left[ \left(\frac{\alpha_s}{4\pi}\right)^2 \{ (1-z)^{\gamma_q} \bar{P}_{ns}^{(1)+}(z) + \bar{B}_2 \delta(1-z) \} \right. \\ & \left. + \left(\frac{\alpha_s}{4\pi}\right)^3 \{ (1-z)^{\gamma_q} \bar{P}_{ns}^{(2)+}(z) + \bar{B}_3 \delta(1-z) \} \right] \end{aligned} \quad (54)$$

where  $P_{qq}^{exp}(z)$  is given in (39) and the resummed residuals  $\bar{P}_{ns}^{(i)+}$ ,  $i = 1, 2$  are related to the exact results for  $P_{ns}^{(i)+}$ ,  $i = 1, 2$ , as follows:

$$\bar{P}_{ns}^{(i)+}(z) = P_{ns}^{(i)+}(z) - B_{1+i} \delta(1-z) + \Delta_{ns}^{(i)+}(z) \quad (55)$$

where

$$\begin{aligned} \Delta_{ns}^{(1)+}(z) = & -4C_F \pi \delta_1 \left\{ \frac{1+z^2}{1-z} - f_q \delta(1-z) \right\} \\ \Delta_{ns}^{(2)+}(z) = & -4C_F (\pi \delta_1)^2 \left\{ \frac{1+z^2}{1-z} - f_q \delta(1-z) \right\} \\ & - 2\pi \delta_1 \bar{P}_{ns}^{(1)+}(z) \end{aligned} \quad (56)$$

and

$$\begin{aligned} \bar{B}_2 = & B_2 + 4C_F \pi \delta_1 f_q \\ \bar{B}_3 = & B_3 + 4C_F (\pi \delta_1)^2 f_q - 2\pi \delta_1 \bar{B}_2. \end{aligned} \quad (57)$$

Here, the constants  $B_i$ ,  $i = 2, 3$  are given by the results in Refs. [25, 26] as

$$\begin{aligned}
B_2 &= 4C_G C_F \left( \frac{17}{24} + \frac{11}{3}\zeta_2 - 3\zeta_3 \right) - 4C_F n_f \left( \frac{1}{12} + \frac{2}{3}\zeta_2 \right) + 4C_F^2 \left( \frac{3}{8} - 3\zeta_2 + 6\zeta_3 \right) \\
B_3 &= 16C_G C_F n_f \left( \frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2^2 + \frac{25}{18}\zeta_3 \right) \\
&\quad + 16C_G C_F^2 \left( \frac{151}{64} + \zeta_2 \zeta_3 - \frac{205}{24}\zeta_2 - \frac{247}{60}\zeta_2^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5 \right) \\
&\quad + 16C_G^2 C_F \left( -\frac{1657}{576} + \frac{281}{27}\zeta_2 - \frac{1}{8}\zeta_2^2 - \frac{97}{9}\zeta_3 + \frac{5}{2}\zeta_5 \right) \\
&\quad + 16C_F n_F^2 \left( -\frac{17}{144} + \frac{5}{27}\zeta_2 - \frac{1}{9}\zeta_3 \right) \\
&\quad + 16C_F^2 n_F \left( -\frac{23}{16} + \frac{5}{12}\zeta_2 + \frac{29}{30}\zeta_2^2 - \frac{17}{6}\zeta_3 \right) \\
&\quad + 16C_F^3 \left( \frac{29}{32} - 2\zeta_2 \zeta_3 + \frac{9}{8}\zeta_2 + \frac{18}{5}\zeta_2^2 + \frac{17}{4}\zeta_3 - 15\zeta_5 \right),
\end{aligned} \tag{58}$$

where  $\zeta_n$  is the Riemann zeta function evaluated at argument  $n$ . The detailed phenomenological consequences of the fully exponentiated 2- and 3-loop DGLAP kernel set will appear elsewhere [21].

In summary, we have used exact re-arrangement of the QCD Feynman series to isolate and resum the leading IR contributions to the physical processes that generate the evolution kernels in DGLAP theory. In this way, we have obviated the need to employ artificial mathematical regularization of the attendant IR singularities as the theory's higher order corrections naturally tame these singularities. The resulting IR-improved anomalous dimension matrix behaves more physically for large  $n$  and receives significant effects at finite  $n$  from the exponentiation.

We in principle can make contact with the moment-space resummation results in Ref. [27] but we stress that these results have necessarily been obtained after making a Mellin transform of the mathematical artifact which we address in this paper. Thus, the results in Ref. [27] do not in any way contradict the analysis in this paper.

We note that the program of improvement of the hadron cross section calculations for LHC physics advanced herein should be distinguished from the results in Refs. [28, 29]. Indeed, recalling the standard hadron cross section formula

$$\sigma = \sum_{i,j} dx_1 dx_2 F_i(x_1) F_j(x_2) \hat{\sigma}(x_1 x_2 s) \tag{59}$$

where  $\{F_i(x)\}$  are the respective parton densities and  $\hat{\sigma}(x_1 x_2 s)$  is the respective reduced hard parton cross section, the resummation results in Refs. [28, 29] address, by summing the large logs in Mellin transform space, the  $x_1 x_2 \rightarrow 1$  limit of  $\hat{\sigma}(x_1 x_2 s)$  whereas the results above address the improvement, by resummation in x-space, of the calculation of the parton densities  $\{F_i(x)\}$  for all values of  $x$ . Thus, the program of improvement

presented above is entirely complementary to that in Refs. [28,29] and both programs of improvement are needed for precision LHC physics.

Finally, we address the issue of the relationship between the re-arrangement that we have made of the exact leading-logs in the QCD perturbation theory and the usual treatment in the non-exponentiated DGLAP theory. If one expands out the exponentiated kernels, using the distribution identity

$$(1-z)^{a-1} = \frac{1}{a}\delta(1-z) + \frac{1}{(1-z)_+} + \sum_{j=1}^{\infty} \frac{a^j}{j!} \left[ \frac{\ln^j(1-z)}{1-z} \right]_+, \quad (60)$$

one can see that for example  $P_{qq}$  and  $P_{qq}^{exp}$  agree to leading order, so that the leading log series which they generate for the respective NS structure functions also agree through leading order in  $\frac{\alpha_s}{\pi}L$  where  $L$  is the respective big log in momentum-space. At higher orders then, we have a different result for the  $\{F_i\}$ , let us denote them by  $\{F'_i\}$ , and a different result for the reduced cross section, let us denote it by  $\hat{\sigma}'$ , such that we get the same perturbative QCD cross section,

$$\begin{aligned} \sigma &= \sum_{i,j} \int dx_1 dx_2 F_i(x_1) F_j(x_2) \hat{\sigma}(x_1 x_2 s) \\ &= \sum_{i,j} \int dx_1 dx_2 F'_i(x_1) F'_j(x_2) \hat{\sigma}'(x_1 x_2 s) \end{aligned} \quad (61)$$

order by order in perturbation theory. The exponentiated kernels are used to factorize the mass singularities from the unfactorized reduced cross section and this generates  $\hat{\sigma}'$  instead of the usual  $\hat{\sigma}$  whose factorized form is generated using the usual DGLAP kernels. We thus have the same leading log series for  $\sigma$  as does the usual calculation with un-exponentiated DGLAP kernels. We have an important advantage: the lack of  $+$ -functions in the generation of the configuration space functions  $\{F'_i, \hat{\sigma}'\}$  means that these functions lend themselves more readily to Monte Carlo realization to arbitrarily soft radiative effects, both for the generation of the parton shower associated to the  $\{F'_i\}$  and for the attendant remaining radiative effects in  $\hat{\sigma}'$ .

Further consequences of our results for LHC physics will be presented elsewhere [21].

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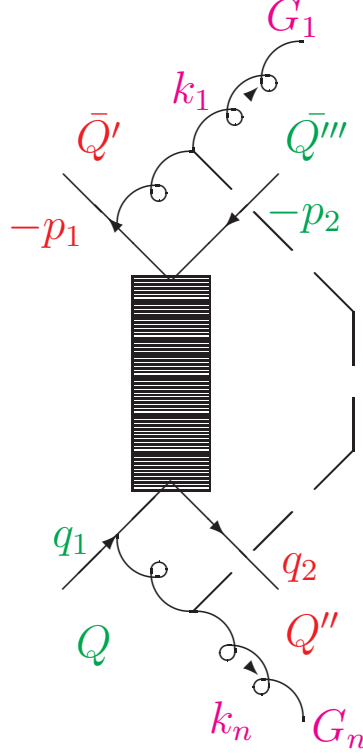


Figure 2: The process  $\bar{Q}'Q \rightarrow \bar{Q}''' + Q'' + n(G)$ . The four-momenta are indicated in the standard manner:  $q_1$  is the four-momentum of the incoming  $Q$ ,  $q_2$  is the four-momentum of the outgoing  $Q''$ , etc., and  $Q = u, d, s, c, b, G$ .

## Appendix

In this Appendix, we present the new QCD exponentiation theory which has been developed in Refs. [16,17] as it is not generally familiar. The goal is to make the current paper self-contained.

For definiteness, we will use the process in Fig. 2,  $\bar{Q}'(p_1)Q(q_1) \rightarrow \bar{Q}'''(p_2)Q''(q_2) + G_1(k_1) \cdots G_n(k_n)$ , as the proto-typical process, where we have written the kinematics as it is illustrated in the figure. This process, which dominates processes such as  $t\bar{t}$  production at FNAL, contains all of the theoretical issues that we must face at the parton level to establish, as an extension of the original ideas of Yennie, Frautschi and Suura (YFS) [18], QCD soft exponentiation by MC methods – applicability to other related processes will be immediate. For reference, let us also note that, in what follows, we use the GPS conventions of Ref. [30] for spinors  $\{u, v, \mathbf{u}\}$  and the attendant photon and gluon polarization vectors that follow therefrom:

$$(\epsilon_\sigma^\mu(\beta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu u_\sigma(\beta)}{\sqrt{2} \bar{u}_{-\sigma}(k)u_\sigma(\beta)}, \quad (\epsilon_\sigma^\mu(\zeta))^* = \frac{\bar{u}_\sigma(k)\gamma^\mu \mathbf{u}_\sigma(\zeta)}{\sqrt{2} \bar{u}_{-\sigma}(k)\mathbf{u}_\sigma(\zeta)}, \quad (62)$$

with  $\beta^2 = 0$  and  $\zeta$  defined in Ref. [30], so that all phase information is strictly known in our amplitudes. This means that, although we shall use the older EEX realization



of YFS MC exponentiation as defined in Ref. [31], the realization of our results via the the newer CEEX realization of YFS exponentiation in Ref. [31] is also possible and is in progress [21].

Specifically, the authors in Refs. [17] have analyzed how in the special case of Born level color exchange one applies the YFS theory to QCD by extending the respective YFS IR singularity analysis to QCD to all orders in  $\alpha_s$ . Here, unlike what was emphasized in Refs. [17], we focus on the YFS theory as a general re-arrangement of renormalized perturbation theory based on its IR behavior, just as the renormalization group is a general property of renormalized perturbation theory based on its UV (ultra-violet) behavior. We will thus keep our arguments entirely general from the outset, so that it will be immediate that our result applies to any renormalized perturbation theory in which the cross section under study is finite.

Let the amplitude for the emission of  $n$  real gluons in our proto-typical subprocess,  $Q^\alpha + \bar{Q}^{\bar{\alpha}} \rightarrow Q^{\prime\prime\gamma} \bar{Q}^{\prime\prime\bar{\gamma}} + n(G)$ , where  $\alpha, \bar{\alpha}, \gamma$ , and  $\bar{\gamma}$  are color indices, be represented by

$$\mathcal{M}_{\gamma\bar{\gamma}}^{(n)\alpha\bar{\alpha}} = \sum_{\ell} M_{\gamma\bar{\gamma}\ell}^{(n)\alpha\bar{\alpha}}, \quad (63)$$

$M_{\ell}^{(n)}$  is the contribution to  $\mathcal{M}^{(n)}$  from Feynman diagrams with  $\ell$  virtual loops. Symmetrization yields

$$M_{\ell}^{(n)} = \frac{1}{\ell!} \int \prod_{j=1}^{\ell} \frac{d^4 k_j}{(2\pi)^4 (k_j^2 - \lambda^2 + i\epsilon)} \rho_{\ell}^{(n)}(k_1, \dots, k_{\ell}), \quad (64)$$

where this last equation defines  $\rho_{\ell}^{(n)}$  as a symmetric function of its arguments  $k_1, \dots, k_{\ell}$ .  $\lambda$  will be our infrared gluon regulator mass for IR singularities; n-dimensional regularization of the 't Hooft-Veltman [32] type is also possible as we shall see.

We now define the virtual IR emission factor  $S_{QCD}(k)$  for a gluon of 4-momentum  $k$ , for the  $k \rightarrow 0$  regime of the respective 4-dimensional loop integration as in (64), such that

$$\lim_{k \rightarrow 0} k^2 \left( \rho_{\gamma\bar{\gamma}1}^{(n)\alpha\bar{\alpha}}(k) |_{\text{leading Casimir contribution}} - S_{QCD}(k) \rho_{\gamma\bar{\gamma}0}^{(n)\alpha\bar{\alpha}} \right) = 0, \quad (65)$$

where we have now introduced the restriction to the leading color Casimir terms at one-loop<sup>3</sup> so that in the expression for the respective one-loop correction  $\rho_1^{(n)}$  and in that for  $S_{QCD}(k)$  given in Refs. [17], only the terms proportional to  $C_F$  should be retained here as we focus on the  $\bar{f}\bar{f} \rightarrow \bar{f}\bar{f}$  case, where  $f$  denotes a fermion. (Henceforth, when we refer to  $k \rightarrow 0$  gluons we are always referring for virtual gluons to the corresponding regime of the 4-dimensional loop integration in the computation of  $M_{\ell}^{(n)}$ .)

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<sup>3</sup>These correspond with maximally non-Abelian terms in Ref. [33] but computed exactly rather than in the eikonal approximation.

In Ref. [17], the respective authors have calculated  $S_{QCD}(k)$  using the running quark masses to regulate its collinear mass singularities, for example; n-dimensional regularization of the 't Hooft-Veltman type is also possible for these mass singularities and we will also illustrate this presently.

We stress that  $S_{QCD}(k)$  has a freedom in it corresponding to the fact that any function  $\Delta S_{QCD}(k)$  which has the property that  $\lim_{k \rightarrow 0} k^2 \Delta S_{QCD}(k) \rho_0^{(n)} = 0$  may be added to it.

Since the virtual gluons in  $\rho_\ell^{(n)}$  are all on equal footing by the symmetry of this function, if we look at gluon  $\ell$ , for example, we may write, for  $k_\ell \rightarrow (0, 0, 0, 0) \equiv O$  while the remaining  $k_i$  are fixed away from  $O$ , the representation

$$\rho_\ell^{(n)} = S_{QCD}(k_\ell) * \rho_{\ell-1}^{(n)}(k_1, \dots, k_{\ell-1}) + \beta_\ell^1(k_1, \dots, k_{\ell-1}; k_\ell) \quad (66)$$

where the residual amplitude  $\beta_\ell^1(k_1, \dots, k_{\ell-1}; k_\ell)$  will now be taken as defined by this last equation. It has two nice properties:

- it is symmetric in its first  $\ell - 1$  arguments
- the IR singularities for gluon  $\ell$  *that are contained in  $S_{QCD}(k_\ell)$*  are no longer contained in it.

We do not at this point discuss the extent to which there are any further remaining IR singularities for gluon  $\ell$  in  $\beta_\ell^1(k_1, \dots, k_{\ell-1}; k_\ell)$ . In an Abelian gauge theory like QED, as has been shown by Yennie, Frautschi and Suura in Ref. [18], there would not be any further such singularities; for a non-Abelian gauge theory like QCD, this point requires further discussion and we will come back to this point presently.

We rather now stress that if we apply the representation (66) again we may write

$$\begin{aligned} \rho_\ell^{(n)} = & S_{QCD}(k_\ell) S_{QCD}(k_{\ell-1}) * \rho_{\ell-2}^{(n)}(k_1, \dots, k_{\ell-2}) \\ & + S_{QCD}(k_\ell) \beta_{\ell-1}^1(k_1, \dots, k_{\ell-2}; k_{\ell-1}) \\ & + S_{QCD}(k_{\ell-1}) \beta_{\ell-1}^1(k_1, \dots, k_{\ell-2}; k_\ell) \\ & + \beta_\ell^2(k_1, \dots, k_{\ell-2}; k_{\ell-1}, k_\ell), \end{aligned} \quad (67)$$

where this last equation serves to define the function  $\beta_\ell^2(k_1, \dots, k_{\ell-2}; k_{\ell-1}, k_\ell)$ . It has two nice properties:

- it is symmetric in its first  $\ell - 2$  arguments and in its last two arguments  $k_{\ell-1}, k_\ell$
- the infrared singularities for gluons  $\ell - 1$  and  $\ell$  that are contained in  $S_{QCD}(k_{\ell-1})$  and  $S_{QCD}(k_\ell)$  are no longer contained in it.

Continuing in this way, with repeated application of (66), we get finally the rigorous, exact

rearrangement of the contributions to  $\rho_\ell^{(n)}$  as

$$\rho_\ell^{(n)} = S_{QCD}(k_1) \cdots S_{QCD}(k_\ell) \beta_0^0 + \sum_{i=1}^{\ell} \prod_{j \neq i} S_{QCD}(k_j) \beta_1^1(k_i) + \cdots + \beta_\ell^\ell(k_1, \dots, k_\ell), \quad (68)$$

where the virtual gluon residuals  $\beta_i^i(k'_1, \dots, k'_i)$  have two nice properties:

- they are symmetric functions of their arguments
- they do not contain any of the IR singularities which are contained in the product  $S_{QCD}(k'_1) \cdots S_{QCD}(k'_i)$ .

Henceforth, we denote  $\beta_i^i$  as the function  $\beta_i$  for reasons of pedagogy. We can not stress too much that (68) is an *exact* rearrangement of the contributions of the Feynman diagrams which contribute to  $\rho_\ell^{(n)}$ ; it involves no approximations. Here also we note that the question of the absolute convergence of these Feynman diagrams from the standpoint of constructive field theory remains open as usual. Yennie, Frautschi and Suura [18] have already stressed that Feynman diagrammatic perturbation theory is non-rigorous from this standpoint. What we do claim is that the relationship between the YFS expansion and the usual perturbative Feynman diagrammatic expansion is itself rigorous even though neither of the two expansions themselves is rigorous.

Introducing (68) into (63) yields a representation similar to that of YFS, and we will call it a “YFS representation”,

$$\mathcal{M}^{(n)} = e^{\alpha_s B_{QCD}} \sum_{j=0}^{\infty} \mathbf{m}_j^{(n)}, \quad (69)$$

where we have defined

$$\alpha_s(Q) B_{QCD} = \int \frac{d^4 k}{(k^2 - \lambda^2 + i\epsilon)} S_{QCD}(k) \quad (70)$$

and

$$\mathbf{m}_j^{(n)} = \frac{1}{j!} \int \prod_{i=1}^j \frac{d^4 k_i}{k_i^2 - \lambda^2 + i\epsilon} \beta_j(k_1, \dots, k_j). \quad (71)$$

We say that (69) is similar to the respective result of Yennie, Frautschi and Suura in Ref. [18] and is not identical to it because we have not proved that the functions  $\beta_i(k_1, \dots, k_i)$  are completely free of virtual IR singularities. What have shown is that they do not contain the IR singularities in the product  $S_{QCD}(k_1) \cdots S_{QCD}(k_i)$  so that  $\mathbf{m}_j^{(n)}$  does not contain the virtual IR divergences generated by this product when it is integrated over the respective 4j-dimensional j-virtual gluon phase space. In an Abelian gauge theory, there are no other possible virtual IR divergences; in the non-Abelian gauge theory that we treat here, such additional IR divergences are possible and are expected; but, the result (69) does have an improved IR divergence structure over (63) in that all of the IR singularities associated with  $S_{QCD}(k)$  are explicitly removed from the sum over the virtual IR improved loop contributions  $\mathbf{m}_j^{(n)}$  to all orders in  $\alpha_s(Q)$ .

Turning now to the analogous rearrangement of the real IR singularities in the differential cross section associated with the  $\mathcal{M}^{(n)}$ , we first note that we may write this cross section as follows according to the standard methods

$$d\hat{\sigma}^n = \frac{e^{2\alpha_s Re B_{QCD}}}{n!} \int \prod_{m=1}^n \frac{d^3 k_m}{(k_m^2 + \lambda^2)^{1/2}} \delta(p_1 + q_1 - p_2 - q_2 - \sum_{i=1}^n k_i) \bar{\rho}^{(n)}(p_1, q_1, p_2, q_2, k_1, \dots, k_n) \frac{d^3 p_2 d^3 q_2}{p_2^0 q_2^0}, \quad (72)$$

where we have defined

$$\bar{\rho}^{(n)}(p_1, q_1, p_2, q_2, k_1, \dots, k_n) = \sum_{color, spin} \left\| \sum_{j=0}^{\infty} m_j^{(n)} \right\|^2 \quad (73)$$

in the incoming  $Q\bar{Q}$ ' cms system and we have absorbed the remaining kinematical factors for the initial state flux, spin and color averages into the normalization of the amplitudes  $\mathcal{M}^{(n)}$  for reasons of pedagogy so that the  $\bar{\rho}^{(n)}$  are averaged over initial spins and colors and summed over final spins and colors. We now proceed in complete analogy with the discussion of  $\rho_\ell^{(n)}$  above.

Specifically, for the functions  $\bar{\rho}^{(n)}(p_1, q_1, p_2, q_2, k_1, \dots, k_n) \equiv \bar{\rho}^{(n)}(k_1, \dots, k_n)$  which are symmetric functions of their arguments  $k_1, \dots, k_n$ , we define first, for  $n = 1$ ,

$$\lim_{|\vec{k}| \rightarrow 0} \vec{k}^2 \left( \bar{\rho}^{(1)}(k) |_{\text{leading Casimir contribution}} - \tilde{S}_{QCD}(k) \bar{\rho}^{(0)} \right) = 0, \quad (74)$$

where the real infrared function  $\tilde{S}_{QCD}(k)$  is rigorously defined by this last equation and is explicitly computed in Refs. [17], wherein we retain here only the terms proportional to  $C_F$  from the result in Ref. [17] ; like its virtual counterpart  $S_{QCD}(k)$  it has a freedom in it in that any function  $\Delta \tilde{S}_{QCD}(k)$  with the property that  $\lim_{|\vec{k}| \rightarrow 0} \vec{k}^2 \Delta \tilde{S}_{QCD}(k) = 0$  may be added to it without affecting the defining relation (74).

We can again repeat the analogous arguments of Ref. [18], following the corresponding steps in (66)-(71) above for  $S_{QCD}$  to get the ‘‘YFS-like’’ result

$$\begin{aligned} d\hat{\sigma}_{\text{exp}} &= \sum_n d\hat{\sigma}^n \\ &= e^{\text{SUM}_{\text{IR}}(QCD)} \sum_{n=0}^{\infty} \int \prod_{j=1}^n \frac{d^3 k_j}{k_j^0} \int \frac{d^4 y}{(2\pi)^4} e^{iy \cdot (p_1 + q_1 - p_2 - q_2 - \sum k_j) + D_{QCD}} \\ &\quad * \bar{\beta}_n(k_1, \dots, k_n) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0} \end{aligned} \quad (75)$$

with

$$\begin{aligned} \text{SUM}_{\text{IR}}(QCD) &= 2\alpha_s Re B_{QCD} + 2\alpha_s \tilde{B}_{QCD}(K_{\text{max}}), \\ 2\alpha_s \tilde{B}_{QCD}(K_{\text{max}}) &= \int \frac{d^3 k}{k^0} \tilde{S}_{QCD}(k) \theta(K_{\text{max}} - k), \end{aligned}$$

$$D_{\text{QCD}} = \int \frac{d^3 k}{k} \tilde{S}_{\text{QCD}}(k) \left[ e^{-iy \cdot k} - \theta(K_{\text{max}} - k) \right], \quad (76)$$

$$\begin{aligned} \frac{1}{2} \bar{\beta}_0 &= d\sigma^{(1\text{-loop})} - 2\alpha_s \text{Re} B_{\text{QCD}} d\sigma_B, \\ \frac{1}{2} \bar{\beta}_1 &= d\sigma^{B1} - \tilde{S}_{\text{QCD}}(k) d\sigma_B, \quad \dots \end{aligned} \quad (77)$$

where the  $\bar{\beta}_n$  are the QCD hard gluon residuals defined above; they are the non-Abelian analogs of the hard photon residuals defined by YFS. Here, for illustration, we have recorded the relationship between the  $\bar{\beta}_n$ ,  $n = 0, 1$  through  $\mathcal{O}(\alpha_s)$  and the exact one-loop and single bremsstrahlung cross sections,  $d\sigma^{(1\text{-loop})}$ ,  $d\sigma^{B1}$ , respectively, where the latter may be taken from Ref. [34] We stress two things about the right-hand side of (75) :

- It does not depend on the dummy parameter  $K_{\text{max}}$  which has been introduced for cancellation of the infrared divergences in  $SUM_{IR}(QCD)$  to all orders in  $\alpha_s(Q)$  where  $Q$  is the hard scale in the parton scattering process under study here.
- Its analog can also be derived in our new CEEX [31] format.

We now return to the property of (75) that distinguishes it from the Abelian result derived by Yennie, Frautschi and Suura – namely, the fact that, owing to its non-Abelian gauge theory origins, it is in general expected that there are infrared divergences in the  $\bar{\beta}_n$  which were not removed into the  $S_{QCD}, \tilde{S}_{QCD}$  when these infrared functions were isolated in our derivation of (75).

More precisely, the left-hand side of (75) is the fundamental reduced parton cross section and it should be infrared finite or else the entire QCD parton model has to be abandoned.

There is an observation in the literature [35] that unless we use the approximation of massless incoming quarks, the reduced parton cross section on the left-hand side of (75) diverges in the infrared regime at  $\mathcal{O}(\alpha_s^2(Q))$ . We do not go into this issue here but either use the quark masses strictly as collinear limit regulators so that they are set to zero in the numerators of all Feynman diagrams in such a way that the limit  $\lim_{m_q^2/E_q^2 \rightarrow 0}$ , where  $E_q$  is the quark energy, is taken everywhere that it is finite or, alternatively, we use n-dimensional methods to regulate such divergences while setting the quark masses to zero as that is an excellent approximation for the light quarks at FNAL and LHC energies – we take this issue up elsewhere.

From the infrared finiteness of the left-hand side of (75) and the infrared finiteness of  $SUM_{IR}(QCD)$ , it follows that the quantity

$$d\bar{\sigma}_{\text{exp}} \equiv e^{-SUM_{IR}(QCD)} d\hat{\sigma}_{\text{exp}}$$

must also be infrared finite to all orders in  $\alpha_s$ .

As we assume the QCD theory makes sense in some neighborhood of the origin for  $\alpha_s$ , we conclude that each order in  $\alpha_s$  must make an infrared finite contribution to  $d\bar{\sigma}_{\text{exp}}$ .

At  $\mathcal{O}(\alpha_s^0(Q))$ , the only contribution to  $d\bar{\sigma}_{\text{exp}}$  is the respective Born cross section given by  $\bar{\beta}_0^{(0)}$  in (75) and it is obviously infrared finite, where we use henceforth the notation  $\bar{\beta}_n^{(\ell)}$  to denote the  $\mathcal{O}(\alpha_s^\ell(Q))$  part of  $\bar{\beta}_n$ . Thus, we conclude that the lowest hard gluon residual  $\bar{\beta}_0^{(0)}$  is infrared finite.

Let us now define the left-over non-Abelian infrared divergence part of each contribution  $\bar{\beta}_n^{(\ell)}$  via

$$\bar{\beta}_n^{(\ell)} = \tilde{\bar{\beta}}_n^{(\ell)} + D\bar{\beta}_n^{(\ell)}$$

where the new function  $\tilde{\bar{\beta}}_n^{(\ell)}$  is now completely free of any infrared divergences and the function  $D\bar{\beta}_n^{(\ell)}$  contains all left-over infrared divergences in  $\bar{\beta}_n^{(\ell)}$  which are of non-Abelian origin and is normalized to vanish in the Abelian limit  $f_{abc} \rightarrow 0$  where  $f_{abc}$  are the group structure constants.

Further, we define  $D\bar{\beta}_n^{(\ell)}$  by a minimal subtraction of the respective IR divergences in it so that it only contains the actual pole and transcendental constants,  $1/\epsilon - C_E$  for  $\epsilon = 2 - d/2$ , where  $d$  is the dimension of space-time, in dimensional regularization or  $\ln \lambda^2$  in the gluon mass regularization. Here,  $C_E$  is Euler's constant.

For definiteness, we write this out explicitly as follows:

$$\int dPh D\bar{\beta}_n^{(\ell)} \equiv \sum_{i=1}^{n+\ell} d_i^{n,\ell} \ln^i(\lambda^2)$$

where the coefficient functions  $d_i^{n,\ell}$  are independent of  $\lambda$  for  $\lambda \rightarrow 0$  and  $dPh$  is the respective n-gluon Lorentz invariant phase space.

At  $\mathcal{O}(\alpha_s^n(Q))$ , the IR finiteness of the contribution to  $d\bar{\sigma}_{\text{exp}}$  then requires the contribution

$$\begin{aligned} d\bar{\sigma}_{\text{exp}}^{(n)} \equiv & \int \sum_{\ell=0}^n \frac{1}{\ell!} \prod_{j=1}^{\ell} \int_{k_j \geq K_{\text{max}}} \frac{d^3 k_j}{k_j} \tilde{S}_{QCD}(k_j) \sum_{i=0}^{n-\ell} \frac{1}{i!} \prod_{j=\ell+1}^{\ell+i} \\ & \int \frac{d^3 k_j}{k_j^0} \bar{\beta}_i^{(n-\ell-i)}(k_{\ell+1}, \dots, k_{\ell+i}) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0} \end{aligned} \quad (78)$$

to be finite.

From this it follows that

$$\begin{aligned} Dd\bar{\sigma}_{\text{exp}}^{(n)} \equiv & \int \sum_{\ell=0}^n \frac{1}{\ell!} \prod_{j=1}^{\ell} \int_{k_j \geq K_{\text{max}}} \frac{d^3 k_j}{k_j} \tilde{S}_{QCD}(k_j) \sum_{i=0}^{n-\ell} \frac{1}{i!} \prod_{j=\ell+1}^{\ell+i} \\ & \int \frac{d^3 k_j}{k_j^0} D\bar{\beta}_i^{(n-\ell-i)}(k_{\ell+1}, \dots, k_{\ell+i}) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0} \end{aligned} \quad (79)$$

is finite. Since the integration region for the final particles is arbitrary, the independent powers of the IR regulator  $\ln(\lambda^2)$  in this last equation must give vanishing contributions.

This means that we can drop the  $D\bar{\beta}_n^{(\ell)}$  from our result (75) because they do not make a net contribution to the final parton cross section  $\hat{\sigma}_{\text{exp}}$ . We thus finally arrive at the new rigorous result

$$\begin{aligned}
d\hat{\sigma}_{\text{exp}} &= \sum_n d\hat{\sigma}^n \\
&= e^{\text{SUM}_{\text{IR}}(\text{QCD})} \sum_{n=0}^{\infty} \int \prod_{j=1}^n \frac{d^3 k_j}{k_j} \int \frac{d^4 y}{(2\pi)^4} e^{iy \cdot (p_1 + q_1 - p_2 - q_2 - \sum k_j) + D_{\text{QCD}}} \\
&\quad * \tilde{\beta}_n(k_1, \dots, k_n) \frac{d^3 p_2}{p_2^0} \frac{d^3 q_2}{q_2^0}
\end{aligned} \tag{80}$$

where now the hard gluon residuals  $\tilde{\beta}_n(k_1, \dots, k_n)$  defined by

$$\tilde{\beta}_n(k_1, \dots, k_n) = \sum_{\ell=0}^{\infty} \tilde{\beta}_n^{(\ell)}(k_1, \dots, k_n) \tag{81}$$

are free of all infrared divergences to all orders in  $\alpha_s(Q)$ . This is a basic result of this Appendix.

We note here that, contrary to what was claimed in the Appendix of the first paper in Refs. [17] and consistent with what is explained in the third reference in [17], the arguments in the first paper in Refs. [17] are not sufficient to derive the respective analog of eq.(80); for, they did not really expose the compensation between the left over genuine non-Abelian IR virtual and real singularities between  $\int dPh\bar{\beta}_n$  and  $\int dPh\bar{\beta}_{n+1}$  respectively that really distinguishes QCD from QED, where no such compensation occurs in the  $\bar{\beta}_n$  residuals for QED.

We point-out that the general non-Abelian exponentiation of the eikonal cross sections in QCD has been proven formally in Ref. [33]. The contact between Ref. [33] and our result (80) is that, in the language of Ref. [33], our exponential factor corresponds to the N=1 term in the exponent of eq.(10) of the latter reference. One also sees immediately the fundamental difference between what we derive in (80) and the eikonal formula in Ref. [33]: our result (80) is an *exact* re-arrangement of the complete cross section whereas the result in eq.(10) of Ref. [33] is an *approximation* to the complete cross section in which all terms that could not be eikonalized and exponentiated have been dropped.

## References

- [1] S.L. Glashow, Nucl. Phys. **22** (1961) 579; S. Weinberg, Phys. Rev. Lett. **19** (1967) 1264; A. Salam, in *Elementary Particle Theory*, ed. N. Svartholm (Almqvist and Wiksells, Stockholm, 1968), p. 367; G. 't Hooft and M. Veltman, Nucl. Phys. **B44**,189 (1972) and **B50**, 318 (1972); G. 't Hooft, *ibid.* **B35**, 167 (1971); M. Veltman, *ibid.* **B7**, 637 (1968);
- [2] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343; H. David Politzer, *ibid.* **30** (1973) 1346; see also , for example, F. Wilczek, in *Proc. 16th International Symposium on Lepton and Photon Interactions, Ithaca, 1993*, eds. P. Drell and D.L. Rubin (AIP, NY, 1994) p. 593, and references therein.
- [3] G. Altarelli and G. Parisi, Nucl. Phys. **B126** (1977) 298; Yu. L. Dokshitzer, Sov. Phys. JETP **46** (1977) 641; L. N. Lipatov, Yad. Fiz. **20** (1974) 181; V. Gribov and L. Lipatov, Sov. J. Nucl. Phys. **15** (1972) 675, 938; see also J.C. Collins and J. Qiu, Phys. Rev. **D39** (1989) 1398 for an alternative discussion of the lowest order DGLAP theory.
- [4] S. Kretzer, H.L. Lai, F.I. Olness, W.K. Tung, MSU-HEP-030101, BNL-NT-03-2, RBRC-325; Phys. Rev. **D69** (2004) 114005, and references therein.
- [5] A.D. Martin, R.G. Roberts, W.J. Stirling and R.S. Thorne, preprint IPPP-04-62, DCPT-04-124, CAVENDISH-HEP-2004-28; Eur. Phys. J. **C39** (2005) 155, and references therein.
- [6] M. Gluck, E. Reya, M. Stratmann and W. Vogelsang, preprint DO-TH-2000-14, RBRC-148, TPR-00-21, Phys. Rev. **D63** (2001) 094005.
- [7] See, for example, Robert S. Thorne, A.D. Martin, R.G. Roberts and W.J. Stirling, CAVENDISH-HEP-05-12, hep-ph/0507015, in *Proceedings of 13th International Workshop on Deep Inelastic Scattering (DIS 05), Madison, Wisconsin, 27 Apr - 1 May 2005*, to appear.
- [8] R.D. Field, *Applications of Perturbative QCD*,(Addison-Wesley Publ. Co., Inc, Redwood City, 1989).
- [9] F. Berends et al., "Z Line Shape", in *Z Physics at LEP 1, v. 1*, CERN-89-08, eds. G. Altarelli, R. Kleiss, and C. Verzegnassi,(CERN, Geneva, 1989) p. 89, and references therein.
- [10] S. Jadach, M. Skrzypek and B.F.L. Ward, Phys. Rev. **D47** (1993) 3733; Phys. Lett. **B257** (1991) 173; in "Z<sup>0</sup> Physics", *Proc. XXVth Rencontre de Moriond, Les Arcs, France, 1990*, ed. J. Tran Thanh Van (Editions Frontieres, Gif-Sur-Yvette, 1990); S. Jadach et al., Phys. Rev. **D44** (1991) 2669.
- [11] S. Jadach and B.F.L. Ward, preprint TPJU 19/89; in *Proc. Brighton Workshop*, eds. N. Dombey and F. Boudjema (Plenum, London, 1990), p. 325. (1991) 577.
- [12] C. Balazs et al., Phys. Lett. **B637** (2006) 235.
- [13] D. Acosta et al., (CDF Collaboration), Phys. Rev. Lett. **95** (2005) 022003.
- [14] S. Albino and R. Ball, *Phys. Lett. B* **513**(2001) 93.



- [15] S.V. Mikhailov, *Phys. Lett. B***431** (1998) 387.
- [16] B.F.L. Ward and S. Jadach, *Acta Phys.Polon.* **B33** (2002) 1543; in *Proc. ICHEP2002*, ed. S. Bentvelsen *et al.*,( North Holland, Amsterdam, 2003 ) p. 275 ; B.F.L. Ward and S. Jadach, *Mod. Phys. Lett.***A14** (1999) 491 ; D. DeLaney *et al.*, *Mod. Phys. Lett.* **A12** (1997) 2425; C. Glosser, S. Jadach, B.F.L. Ward and S.A.,*Mod. Phys. Lett.***A 19**(2004) 2113; B.F.L. Ward, C. Glosser, S. Jadach and S.A. Yost, in *Proc. DPF 2004*, Int. J. Mod. Phys. **A20** (2005) 3735; in *Proc. ICHEP04, vol. 1*, eds. H. Chen *et al.*,(World. Sci. Publ. Co., Singapore, 2005) p. 588; B.F.L. Ward and S. Yost, preprint BU-HEPP-05-05, and references therein.
- [17] D. DeLaney *et al.*,*Phys. Rev. D***52** (1995) 108; *Phys. Lett.* **B342** (1995) 239; *Phys. Rev. D***66** (2002) 019903(E).
- [18] D. R. Yennie, S. C. Frautschi, and H. Suura, *Ann. Phys.* **13** (1961) 379; see also K. T. Mahanthappa, *Phys. Rev.* **126** (1962) 329, for a related analysis.
- [19] R.K. Ellis *et al.*, *Phys. Lett.***B78** (1978) 281; *Nucl.Phys.* **B152** (1979) 285; D. Amati, R. Petronzio and G. Veneziano, *ibid.***B146** (1978) 29; S. Libby and G. Sterman, *Phys. Rev. D***18** (1978) 3252; A. Mueller, *ibid.* **D18** (1978) 3705.
- [20] S. G. Gorishnii, S.A. Larin, and F. V. Tkachov, *Phys. Lett.***B124** (1983) 217; S. G. Gorishnii and S.A. Larin, *Nucl. Phys. B***283** (1987) 452.
- [21] B.F.L. Ward, to appear.
- [22] H. Georgi and H. D. Politzer, *Phys. Rev. D***9** (1974) 416; D. Gross and F. Wilczek, *ibid.***9** (1974) 980.
- [23] S. Bethke, *Nucl. Phys. Proc. Suppl.***135** (2004) 345.
- [24] T. Carli *et al.*, in *Proc. HERA-LHC Workshop*, CERN-2005-014, eds. A. De Roeck and H. Jung,(CERN, Geneva, 2005) p. 78.
- [25] E.G. Floratos, D.A. Ross, C. T. Sachrajda, *Nucl.Phys.* **B129**(1977) 66;*ibid.***B139**(1978) 545; *ibid.***B152** (1979) 493,1979; A. Gonzalez-Arroyo, C. Lopez and F.J. Yndurain, *Nucl. Phys.***B153** (1979) 161; A. Gonzalez-Arroyo and C. Lopez, *Nucl. Phys.* **B166** (1980) 429; G. Curci, W. Furmanski and R. Petronzio, *Nucl. Phys.* **B175** (1980) 27; W. Furmanski and R. Petronzio, *Phys. Lett.* **B97** (1980) 437; E.G. Floratos, C. Kounnas and R. Lacaze, *Nucl. Phys.***B192**:417,1981 ; R. Hamberg and W. Van Neerven, *Nucl. Phys.* **B379** (1992) 143.
- [26] S. Moch, J.A.M. Vermaseren and A. Vogt, *Nucl. Phys. B*688 (2004) 101; *ibid.* **B691** (2004) 129, and references therein.
- [27] G. Corcella and L. Magnea, preprint CERN-PH-TH-2005-090, DFTT-15-2005, Jun 2005; hep-ph/0506278, and references therein.
- [28] G. Sterman,*Nucl. Phys.B* **281**, 310 (1987).
- [29] S. Catani and L. Trentadue, *Nucl. Phys.B* **327**, 323 (1989); *ibid.* **353**, 183 (1991).

- [30] S. Jadach, B.F.L. Ward and Z. Was, *Eur. Phys. J.* **C22** (2001) 423.
- [31] S. Jadach, B.F.L. Ward and Z. Was, *Phys. Rev.* **D63** (2001) 113009.
- [32] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44** (1972) 189 and **B50** (1972) 318, and references therein.
- [33] J.G.M. Gatherall, *Phys. Lett.* **B133** (1983) 90.
- [34] R. K. Ellis and J.C. Sexton, *Nucl. Phys.* **B269** (1986) 445.
- [35] C. Di'Lieto, S. Gendron, I.G. Halliday and Christopher T. Sachrajda, *Nucl. Phys.* **B183** (1981) 223; S. Catani *et al.*, *ibid.* **B264** (1986) 588 ; S. Catani, *Z. Phys.* **C37** (1988) 357, and references therein.